Solutions to Practice Exam 2

Tuesday, March 10, 2015

1. Calculate derivatives:

   (a) $x \ln(\cos(x))$

   $$\ln(\cos(x)) - x \cdot \frac{1}{\cos(x)} \sin(x)$$

   (b) $\dfrac{x^2 - 2x + 3}{x - 7}$

   $$\frac{(x-7)(2x-2) - (x^2 - 2x + 3)}{(x-7)^2}$$

   (c) $3\tan^2(x) + 4\cos^2(x)$

   $$6\tan(x)\sec^2(x) - 8\cos(x)\sin(x)$$

   (d) $\sqrt{5x} - \sqrt{3x}$

   $$\frac{1}{2\sqrt{5x}} \cdot 5 - \frac{1}{2\sqrt{3x}} \cdot 3$$

   (e) 4^x

   $$4^x \ln(4)$$

2. Prove that

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

using implicit differentiation and your knowledge of trigonometry.

We want to find dy/dx, where

$$y = \tan^{-1}(x).$$

Rewrite this as

$$x = \tan(y).$$

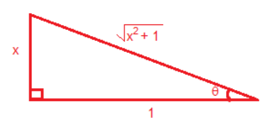
Take the derivative of each side with respect to x:

$$1 = sec^2(y)\frac{dy}{dx}$$

Solve for dy/dx:

$$\frac{dy}{dx} = \cos^2(y) = \cos^2(\tan^{-1}(x))$$

If $\tan\theta = x$, then (up to scaling) we're looking at this triangle:



So we see that

$$\frac{dy}{dx} = \cos^2(\tan^{-1}(x)) = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 = \frac{1}{1+x^2}.$$

3. Sketch a graph of the function

$$f(x) = \frac{x}{x^2+3},$$

indicating any and all local extrema, inflection points, and/or asymptotes the function may have.

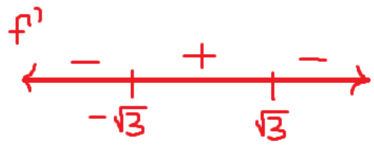
We have

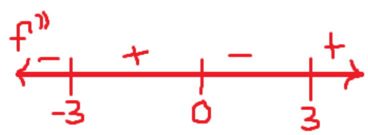
$$f'(x) = \frac{(x^2+3) - x(2x)}{(x^2+3)^2} = \frac{x^2+3-2x^2}{(x^2+3)^2} = \frac{3-x^2}{(x^2+3)^2}$$

and

$$\begin{aligned} f''(x) &= \frac{(x^2+3)^2(-2x) - (3-x^2)2(x^2+3)(2x)}{(x^2+3)^4} = \frac{(x^2+3)(-2x) - (3-x^2)2(2x)}{(x^2+3)^3} \\ &= \frac{(-2x^3-6x) - (12x-4x^3)}{(x^2+3)^3} = \frac{2x^3-18x}{(x^2+3)^3} = \frac{2x(x^2-9)}{(x^2+3)^3} = \frac{2x(x-3)(x+3)}{(x^2+3)^3} \end{aligned}$$

so the critical points are $x = \pm\sqrt{3}$ and there are potential inflection points at $x = \pm 3$ and $x = 0$.

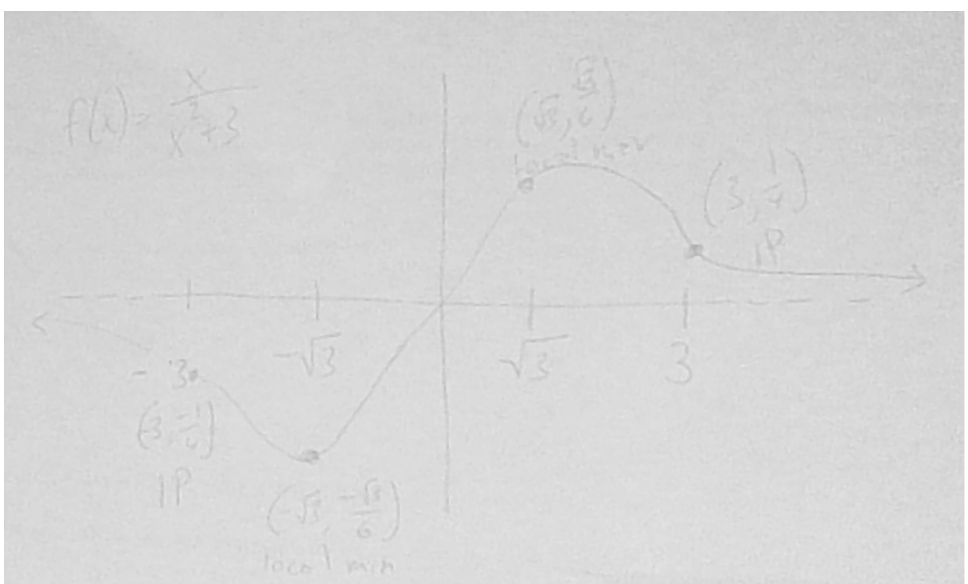




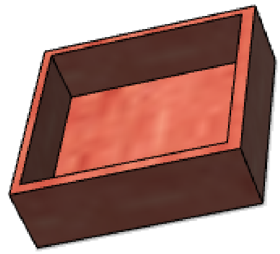
We have $f(-3) = -1/4$, $f(-\sqrt{3}) = -\sqrt{3}/6$, $f(0) = 0$, $f(\sqrt{3}) = \sqrt{3}/6$, and $f(3) = 1/4$. Finally,

$$\lim_{x\to\infty} \frac{x}{x^2+3} = \lim_{x\to\infty} \frac{1}{x + \frac{3}{x}} = 0$$

and similarly for the limit as $x \to -\infty$. So the graph looks like



4. You want to construct an open-topped cardboard box with a square base, like so:



If the box needs to have a volume of four cubic meters, how many square meters of cardboard do you need to buy? (Don't worry about taking the thickness of the cardboard into account).

Say that the base is $s \times s$ and the height is h. Then we want to minimize $A = s^2 + 4sh$ subject to the constraint $V = s^2 h = 4\ \text{m}^3$. Solving for h, we have

$$h = \frac{4\ \text{m}^3}{s^2}, \qquad \text{so} \qquad A = s^2 + 4s\left(\frac{4\ \text{m}^3}{s^2}\right) = s^2 + \frac{16\ \text{m}^3}{s}$$

Taking the derivative with respect to s, we get

$$A'(s) = 2s + \frac{-16\ \text{m}^3}{s^2}$$

At a minimum, we know that the derivative will be zero (since a minimum is a point at which the function stops decreasing – negative derivative – and starts increasing – positive derivative). So we should figure out where $A'(s) = 0$:

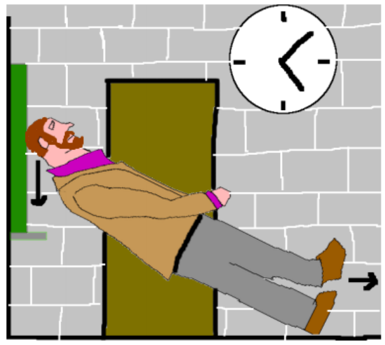
$$2s - \frac{16\ \text{m}^3}{s^2} = 0 \quad \Rightarrow \quad 2s = \frac{16\ \text{m}^3}{s^2} \quad \Rightarrow \quad 2s^3 = 16\ \text{m}^3 \quad \Rightarrow \quad s = 2\ \text{m}$$

So we want the box to be $2\ \text{m} \times 2\ \text{m} \times 1\ \text{m}$. That means that we require

$$(2\ \text{m})^2 + 4(2\ \text{m})(1\ \text{m}) = 4\ \text{m}^2 + 8\ \text{m}^2 = 12\ \text{m}^2$$

of cardboard.

5. A five-foot-tall man leans against a wall to take a nap. After a while, his feet start sliding away from the wall at a rate of one inch per minute. His head remains against the wall, and his body remains rigid, like so:



How fast is his head sliding down the wall when its four feet above the ground?

By the Pythagorean theorem,

$$x^2 + y^2 = (5\text{ ft})^2 = 25\text{ ft}^2;$$

differentiating each side with respect to time,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Also, when $y = 4$ ft, we see that

$$x = \sqrt{25\text{ ft}^2 - (4\text{ ft})^2} = \sqrt{9\text{ ft}^2} = 3\text{ ft}.$$

So we have

$$2(3\text{ ft})\left(1\,\frac{\text{in}}{\text{min}}\right) + 2(4\text{ ft})\frac{dy}{dt} = 0.$$

Solving,

$$\frac{dy}{dt} = \frac{-2(3\text{ ft})\left(1\,\frac{\text{in}}{\text{min}}\right)}{2(4\text{ ft})} = \frac{3}{4}\text{ in/min}.$$